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PARTITION THEOREMS AND ULTRAFILTERS

BY

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ABSTRACT. We introduce a class of ultrafilters on ω called k-arrow ultrafilters and characterized by the partition relation $U \to (U, k)^2$. These are studied in conjunction with P-points, Q-points, weakly Ramsey and Ramsey ultrafilters.

1. **Introduction**. Unless otherwise specified, we will use the term ultrafilter to mean a proper nonprincipal ultrafilter on ω. We will be interested in the following types of ultrafilters U, the first four of which are well known.

1. U is a *P-point* iff for every partition $\{A_n: n \in \omega\}$ of ω with $A_n \notin U$ for each n, there exists a set $X \in U$ such that $X \cap A_n$ is finite for each $n \in \omega$.

2. U is a *Q-point* iff for every partition $\{A_n: n \in \omega\}$ of ω with A_n finite for each n, there exists a set $X \in U$ such that $|X \cap A_n| \leqslant 1$ for each $n \in \omega$.

3. U is a *Ramsey ultrafilter* iff for every function f: $[\omega]^2 \to 2$ there exists a set $X \in U$ such that $|f([X]^2)| = 1$.

4. U is a *weakly Ramsey ultrafilter* iff for every function f: $[\omega]^2 \to 3$ there exists a set $X \in U$ such that $|f([X]^2)| \leqslant 2$.

5. U is a *k-arrow ultrafilter* iff $3 \leqslant k < \omega$, and for every function f: $[\omega]^2 \to 2$, either there exists a set $X \in U$ such that $f([X]^2) = \{0\}$, or else there exists a set $Y \in [\omega]^k$ such that $f([Y]^2) = \{1\}$. U is an *arrow ultrafilter* iff U is a k-arrow ultrafilter for each k with $3 \leqslant k < \omega$.

An ultrafilter which is both a k-arrow ultrafilter and a P-point (Q-point) will be called a *k-arrow P-point* (*k-arrow Q-point*). Similarly, we will speak of *arrow P-points* and *arrow Q-points*.

It is well known [3] that an ultrafilter is Ramsey iff it is both a P-point and a Q-point. Moreover, it is clear that every Ramsey ultrafilter is weakly Ramsey, and it will be shown later (Corollary 2.5) that every weakly Ramsey ultrafilter is an arrow P-point. If U is a $(k + 1)$-arrow ultrafilter then clearly U is a k-arrow ultrafilter.

In §2 we prove several characterizations of these classes of ultrafilters. §3 contains the main results. We show that if U is weakly Ramsey and V is a k-arrow P-point, and U and V are incompatible in the Rudin-Keisler ordering, then $V \times U$ is a k-arrow ultrafilter. It is also shown that most of

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these hypotheses are necessary. In §4 we give several constructions (most assuming a consequence of Martin's axiom) showing that the classes of ultrafilters being considered are distinct. In §5 we briefly consider extensions to the case where U is a κ-complete ultrafilter on the uncountable measurable cardinal κ. §6 contains some questions.

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2. Some characterizations. Recall that the *Rudin-Keisler ordering on ultrafilters* is given by $D \leqslant_{RK} U$ iff there is a function $f: \omega \to \omega$ such that $X \in D$ iff $f^{-1}(X) \in U$. If $D \leqslant_{RK} U$ and $U \leqslant_{RK} D$, then we say that U and D are *isomorphic* (denoted $U \cong D$). It is straightforward to verify that, except for Q-points, the classes of ultrafilters that we are considering are closed downward in the Rudin-Keisler ordering.

It is well known that an ultrafilter U is Ramsey iff it is minimal with respect to the Rudin-Keisler ordering. Several other equivalent definitions of a Ramsey ultrafilter are also known (see [3], [4], and [11]). In view of our definition of k-arrow ultrafilters, it seems appropriate to add a few more equivalents of "Ramseyness" to the list. First, however, a bit of notation should be introduced.

NOTATION. If P, Q and R are collections of sets and $n \in \omega$, then $P \to (Q, R)^n$ denotes the assertion that whenever $Z \in P$ and $f: [Z]^n \to 2$, either there exists a set $X \in Q$ such that $f([X]^n) = \{0\}$, or there exists a set $Y \in R$ such that $f([Y]^n) = \{1\}$. In order to conform with the customary notation, we will let k denote the collection of all sets of cardinality k and we will write $P \to (Q)^n_2$ in case $Q = R$. For example, U is Ramsey iff $U \to (U)^2_2$, and U is a k-arrow ultrafilter iff $U \to (U, k)^2$.

THEOREM 2.1. *If U is an ultrafilter on ω, then the following are equivalent conditions on U.*

(i) *U is a Ramsey ultrafilter.*

(ii) $U \to (U, \omega)^2$.

(iii) $U \to (U, 4)^3$.

PROOF. It is well known [3] that if U is Ramsey then $U \to (U)^n_2$ for every $n \in \omega$. Hence, (i) immediately implies both (ii) and (iii). We will show that (ii) $\to$ (i) and (iii) $\to$ (i).

(ii) $\to$ (i): It suffices to show that if $U \to (U, \omega)^2$ then U is both a P-point and a Q-point. Suppose first that U is not a P-point and let $\{A_n: n \in \omega\}$ be a partition of ω such that $A_n \notin U$ and such that for every $X \in U$ there exists some $n \in \omega$ such that $|X \cap A_n| = \omega$. Notice that we lose no generality in assuming that $|A_n| = \omega$ for all $n \in \omega$. Moreover, we actually have that if

$X \in U$ then $|X \cap A_n| = \omega$ for infinitely many $n \in \omega$. Define a relation $\otimes$ on ω by

$$n \otimes m \text{ iff } \begin{cases} n \in A_i \text{ and } m \in A_j \text{ and } i < j, \text{ or} \\ n, m \in A_i \text{ and } n < m. \end{cases}$$

Notice that $\otimes$ is a well ordering. Define f: $[\omega]^2 \to 2$ by

$$f(\{n, m\}) = 0 \quad \text{iff} < \restriction \{n, m\} = \otimes \restriction \{n, m\}.$$

If $f([X]^2) = \{1\}$ then $(X, < \restriction X) = (X, \otimes \restriction X)$, so $|X| < \omega$ since there are no infinite descending $\otimes$-chains. On the other hand, if $f([X]^2) = \{0\}$ then $(X, < \restriction X) = (X, \otimes \restriction X)$, so the order type of $(X, \otimes \restriction X)$ is at most ω. Hence, if $|X \cap A_n| = \omega$ for some $n \in \omega$, then $X \cap A_m = 0$ for all $m > n$, and so $X \subseteq \bigcup \{A_i: i \leqslant n\}$ which is not in U. If $|X \cap A_n| < \omega$ for all $n \in \omega$ then our assumptions on the partition $\{A_n: n \in \omega\}$ again force us to conclude that $X \notin U$. Thus, if U is not a P-point, then $U \not\to (U, \omega)^2$.

Suppose now that U is not a Q-point and $\{A_n: n \in \omega\}$ is a partition of ω into finite sets such that for all $X \in U$ there exists some $n \in \omega$ such that $|X \cap A_n| > 1$. Define f: $[\omega]^2 \to 2$ by $f(\{n, m\}) = 1$ iff there exists $i \in \omega$ such that $n, m \in A_i$. If $f([X]^2) = \{1\}$ then $X \subseteq A_i$ for some $i \in \omega$, so X is finite. If $f([X]^2) = \{0\}$ then $|X \cap A_i| \leqslant 1$ for all $i \in \omega$, so $X \notin U$. Thus $U \not\to (U, \omega)^2$, which was to be shown.

(iii) $\to$ (i): Suppose $U \to (U, 4)^3$ and let f: $[\omega]^2 \to 2$. Define g: $[\omega]^3 \to 2$ by $g(\{x, y, z\}) = 1$ iff $f(\{x, y\}) = 0$ and $f(\{y, z\}) = 1$, where $x < y < z$. It is clearly impossible to have $g([Y]^3) = \{1\}$ for any set $Y \in [\omega]^4$. Hence, there exists a set $X \in U$ such that $g([X]^3) = \{0\}$. If $f([X]^2) = \{1\}$ then we are clearly done. Suppose then that $f(\{x, y\}) = 0$ for some pair $\{x, y\} \in [X]^2$ with $x < y$. Let $X' = \{n \in X: n > y\}$. Then $X' \in U$ and we claim that $f([X']^2) = \{0\}$. If not, then we have $x, y, z, w \in X$ with $x < y < z < w$ and $f(\{x, y\}) = 0$ and $f(\{z, w\}) = 1$. If $f(\{y, z\}) = 1$ then $g(\{x, y, z\}) = 1$, and if $f(\{y, z\}) = 0$ then $g(\{y, z, w\}) = 1$. Hence either case contradicts the fact that $g([X]^3) = \{0\}$. This completes the proof.

In order to give some characterizations of P-points and weakly Ramsey ultrafilters we will need the following definitions.

Definition 2.2. (i) If f: $[\omega]^2 \to k$, U is an ultrafilter, and $j < k < \omega$, then f is a *j-partition for U* iff $\{x: \{y: f(\{x, y\}) = j\} \in U\} \in U$.

(ii) If f: $[\omega]^2 \to k$ then a set $X \subseteq \omega$ is *eventually homogeneous for f* iff X is infinite and there is a nondecreasing function g: $\omega \to \omega$ such that f is constant on $\{\{p, q\} \in [X]^2: g(p) \leqslant q\}$.

(iii) If $T \subseteq [\omega]^\omega$ then a set $X \subseteq \omega$ is *eventually homogeneous for T* iff X is infinite and there is a nondecreasing function g: $\omega \to \omega$ such that $[X]^\omega_g \subseteq T$ or $[X]^\omega_g \cap T = 0$, where

$$[X]^\omega_g = \{Y \in [X]^\omega : \forall y \in Y,\ \{y+1, \ldots, g(y)-1\} \cap Y = 0\}.$$

REMARKS. (i) If f: $[\omega]^2 \to k$ and $k < \omega$, then f is a j-partition for U for some unique $j < k$.

(ii) If g: $\omega \to \omega$ is given by $g(n) = n + 1$ then $[X]^\omega_g = [X]^\omega$.

(iii) If $T \subseteq [\omega]^\omega$ and there exists an eventually homogeneous set for T, then T is in fact *Ramsey* (i.e. there exists a set $Y \in [\omega]^\omega$ such that either $[Y]^\omega \subseteq T$ or $[Y]^\omega \cap T = 0$).

THEOREM 2.3. *If U is an ultrafilter on ω then the following are equivalent conditions on U.*

(i) *U is a P-point.*

(ii) *If $T \subseteq [\omega]^\omega$ and T is Σ^1_1 then there exists a set $X \in U$ such that X is eventually homogeneous for T.*

(iii) *If $k \in \omega$ and f: $[\omega]^2 \to k$, then there exists a set $X \in U$ such that X is eventually homogeneous for f.*

PROOF. The proof that (i) $\to$ (ii) is based on an adaptation of Ellentuck's techniques [5] as employed by Milliken [12]. We will not give the proof here since it is quite lengthy and we will not be needing the fact that P-points satisfy condition (ii). The reader is thus referred to [17] for the proof that (i) $\to$ (ii). In view of this, we will give the proof that (i) $\to$ (iii) even though (iii) follows immediately from (ii).

(i) $\to$ (iii): Suppose U is a P-point and f: $[\omega]^2 \to k$. Choose $j < k$ such that f is a j-partition for U. Then there exists a set $B \in U$ such that for all $x \in B$ we have $A_x \in U$ where $A_x = \{y: f(\{x, y\}) = j\}$. If g is any nondecreasing function mapping ω to ω such that $X - A_x \subseteq g(x)$ for every $x \in X$, then g shows that X is eventually homogeneous for f.

(iii) $\to$ (i): Suppose $\{A_i: i \in \omega\}$ is a partition of ω such that for each $i \in \omega$, $A_i \not\in U$. Define f: $[\omega]^2 \to 2$ by $f(\{m, n\}) = 0$ iff there exists an $i \in \omega$ such that $m, n \in A_i$. It is easy to see that if $X \in U$ is eventually homogeneous for f then $X \cap A_i$ is finite for each $i \in \omega$. Hence U is a P-point and the proof is complete.

THEOREM 2.4. *If U is an ultrafilter on ω then the following are equivalent conditions on U.*

(i) *U is a weakly Ramsey ultrafilter.*

(ii) *For every partition $\{A_n: n \in \omega\}$ of ω with $A_n \notin U$ and for every function f: $[\omega]^2 \to 2$ there exists a set $X \in U$ such that $|X \cap A_n| < \omega$ for all $n \in \omega$ and f is constant on $[X \cap A_n]^2$ for all $n \in \omega$.*

(iii) *If $n \in \omega$ and f_i: $[\omega]^2 \to 2$ is a 0-partition for U for each $i < n$ then either* (a) *or* (b) *holds*:

(a) $\exists X \in U \exists i < n\ (f_i([X]^2) = \{0\})$.

(b) $\forall k \in \omega \exists X \in [\omega]^k \forall i < n\, (f_i([X]^2) = \{1\})$.

(iv) *If* f_0, f_1: $[\omega]^2 \to 2$ *are* 0-*partitions for* U *then either* (a′) *or* (b′) *holds*:

(a′) $\exists X \in U \exists i < 2\, (f_i([X]^2) = \{0\})$.

(b′) $\exists \{p, q\} \forall i < 2\, (f_i(\{p, q\}) = 1)$.

REMARK. The authors originally worked with condition (ii). It was Kanamori who pointed out to us that (ii) is in fact equivalent to the more familiar notion (i). The proof that (i) is equivalent to (ii) is included with his kind permission.

PROOF. (i) → (ii): Suppose that U is weakly Ramsey and let $\{A_n\colon n \in \omega\}$ and f: $[\omega]^2 \to 2$ be as in (ii). Since U must be a P-point (see [1]) we can assume without loss of generality that each A_n is finite. Define g: $[\omega]^2 \to 3$ by

$$g(\{x, y\}) = \begin{cases} f(\{x, y\}) & \text{if } \exists n\, [\{x, y\} \subseteq A_n], \\ 2 & \text{otherwise.} \end{cases}$$

Since U is weakly Ramsey, there exists a set $X \in U$ and a set $B \subseteq 3$ such that $|B| \leqslant 2$ and $g([X]^2) \subseteq B$. It is clear that we must have $2 \in B$. Thus, either $1 \notin B$ (yielding $f([X \cap A_n]^2) = \{0\}$) or $0 \notin B$ (yielding $f([X \cap A_n]^2) = \{1\}$).

(ii) → (iii): If U satisfies (ii) then clearly U is a P-point. Suppose now that $n \in \omega$ and for each $i \in n$ we have a 0-partition f_i: $[\omega]^2 \to 2$ for U. Let $X_i \in U$ be eventually homogeneous for f_i and let g_i: $\omega \to \omega$ be the associated function. Let $X = \bigcap \{X_i\colon i < n\}$. Notice that since each f_i is a 0-partition for U it must be the case that $f_i(\{p, q\}) = 0$ whenever $p, q \in X_i$ and $q \geqslant g_i(p)$. We construct disjoint finite sets $\{B_m\colon m \in \omega\}$ such that $B_m \subseteq X$. Let $B_0 = \{\inf(X)\}$. Given B_m let

$$k_m = \max\{ g_i(j)\colon i < n, j \in B_0 \cup \cdots \cup B_m \}$$

and let

$$B_{m+1} = X \cap (k_m + 1) - (B_0 \cup \cdots \cup B_m).$$

Without any loss of generality, we can assume that $\bigcup \{B_{2m}\colon m \in \omega\} \in U$. Now, for each $i < n$ choose $Y_i \in U$ such that f_i is constant on $\bigcup_{m \in \omega} [Y_i \cap B_{2m}]^2$ and let $Z = X \cap Y_0 \cap \cdots \cap Y_{n-1}$. If some $X \cap Y_j$ is homogeneous for some f_i then we are done. If not, then each f_i is constantly one on $[Y_j \cap B_{2m}]^2$ for every m. If $|Z \cap B_{2m}| \geqslant k$ for some k then (b) is true. If not, then there exists $Z' \subseteq Z$ such that $Z' \in U$ and $|Z' \cap B_{2m}| \leqslant 1$ for all m. But then every f_i would be constantly zero on $[Z']^2$.

(iii) → (iv): Trivial.

(iv) → (i): Suppose f: $[\omega]^2 \to 3$ and assume, without loss of generality, that f is a 0-partition for U. Define f_0 and f_1 mapping $[\omega]^2$ to 2 as follows:

$$f_0(\{p,q\}) = \begin{cases} 0 & \text{if } f(\{p,q\}) \in \{0,1\}, \\ 1 & \text{if } f(\{p,q\}) = 2, \end{cases}$$

$$f_1(\{p,q\}) = \begin{cases} 0 & \text{if } f(\{p,q\}) \in \{0,2\}, \\ 1 & \text{if } f(\{p,q\}) = 1. \end{cases}$$

Clearly both f_0 and f_1 are also 0-partitions for U. We clearly cannot have both $f_0(\{p,q\}) = 1$ and $f_1(\{p,q\}) = 1$, so condition (iv) guarantees that $f_i([X]^2) = \{0\}$ for some $X \in U$ and some $i < 2$. If $f_0([X]^2) = \{0\}$ then $f([X]^2) \subseteq \{0,1\}$, and if $f_1([X]^2) = \{0\}$ then $f([X]^2) \subseteq \{0,2\}$. Thus, in either case we have $|f([X]^2)| < 2$ and so U is seen to be a weakly Ramsey ultrafilter.

COROLLARY 2.5. *If U is a weakly Ramsey ultrafilter then U is an arrow P-point.*

PROOF. This follows immediately from the case $n = 1$ of Theorem 2.4(iii).

3. Products and sums that are arrow ultrafilters. Our primary concern in this section is with the problem of determining when a product of ultrafilters is a k-arrow ultrafilter. We begin with some definitions.

DEFINITION 3.1. Let U be an ultrafilter on $\omega \times \omega$ and let D and U_i for $i \in \omega$ be ultrafilters on ω. Then U is the *D-sum of the U_i's* (denoted $U = D \sum U_i$) iff U satisfies the following:

$$U = \{A \subseteq \omega \times \omega: \{x: \{y: (x,y) \in A\} \in U_x\} \in D\}.$$

In the special case where the ultrafilters U_i are all equal to the same ultrafilter V on ω, we have that $D \sum U_i = D \times V$. That is;

$$D \times V = \{A \subseteq \omega \times \omega: \{x: \{y: (x,y) \in A\} \in V\} \in D\}.$$

DEFINITION 3.2. If U and V are ultrafilters on ω then U and V are *incompatible* (in the Rudin-Keisler ordering) iff $U' \not\cong V'$ whenever $U' \leqslant_{RK} U$ and $V' \leqslant_{RK} V$.

The reason that we consider Definition 3.2 is illustrated by the following theorem which gives some conditions on ultrafilters U and V that are necessary if $U \times V$ is to be a k-arrow ultrafilter.

THEOREM 3.3. *Suppose $D \times U$ is a k-arrow ultrafilter. Then D and U are incompatible k-arrow P-points.*

PROOF. Suppose first that D and U are compatible. Thus, there exist ultrafilters D' and U' such that $D' \leqslant_{RK} D$, $U' \leqslant_{RK} U$ and $D' \cong U'$. It is easy to check that $D' \times U' \leqslant_{RK} D \times U$, and, as remarked earlier, the k-arrow ultrafilters are closed downward in the Rudin-Keisler ordering. Hence, if $D \times U$ is a k-arrow ultrafilter then so is $D' \times U'$. We show that this is impossible. Since $D' \cong U'$ it is easy to see that $D' \times U' \cong D' \times D'$. Thus, it suffices to show that $D' \times D' \nrightarrow (D' \times D', 3)^2$. Define f: $[\omega \times \omega]^2 \to 2$ by

$$f(\{(n_1, n_2), (n_3, n_4)\}) = 1 \quad \text{iff } n_1 < n_2 = n_3 < n_4.$$

It is clear that if $f([X]^2) = \{1\}$ then $|X| < 3$. Suppose now that $P \in D' \times D'$. For each $n \in \omega$ let $A_n = \{y\colon (n, y) \in P\}$ and let $B = \{n\colon A_n \in D'\}$. Then $B \in D'$ since $P \in D' \times D'$. Choose $n_1 \in B$ and choose $n_2 \in A_{n_1} \cap B$ such that $n_2 > n_1$. Let $n_3 = n_2$ and choose $n_4 \in A_{n_3}$ such that $n_4 > n_3$. Then

$$\{(n_1, n_2), (n_3, n_4)\} \in [P]^2 \quad \text{and} \quad f(\{(n_1, n_2), (n_3, n_4)\}) = 1.$$

Thus $f([P]^2) \neq \{0\}$ for any $P \in D' \times D'$. This completes the proof that D and U must be incompatible.

The fact that both D and U must be k-arrow ultrafilters follows because D and U are RK-below $D \times U$ and the k-arrow ultrafilters are closed downward in the Rudin-Keisler ordering.

It remains only to show that D and U must both be P-points. Suppose D is not a P-point, and let $\{A_n\colon n \in \omega\}$ be a disjoint partition of ω into infinite sets not in D, such that for all $X \in D$, $\{n\colon X \cap A_n$ is infinite$\}$ is infinite. (This is easily done since D is not a P-point.) Moreover, we can assume that $A_n \cap n = 0$. Define f: $[\omega \times \omega]^2 \to 2$ as follows:

$$f(\{(y, z), (y', z')\}) = 1 \quad \text{iff} \quad y \in A_x \text{ and } y' \in A_{x'} \text{ and } y < x' < z < y'.$$

Suppose first that f is one on all pairs from $\{(y, z), (y', z'), (y'', z'')\}$ where $y < y' < y''$. Then $y \in A_x$, $y' \in A_{x'}$, $y'' \in A_{x''}$ and $y < x' < z < y'$ (since $f(\{(y, z), (y', z')\}) = 1$), and $y' < x'' < z' < y''$ (since $f(\{(y', z'), (y'', z'')\}) = 1$). Thus $z < x''$. But since $f(\{(y, z), (y'', z'')\}) = 1$, we have $y < x'' < z < y''$, so $x'' < z$. This contradiction shows that if $f([X]^2) = \{1\}$ then $|X| < 3$. Suppose now that $P \in D \times U$ and $f([P]^2) = \{0\}$. Let

$$B = \{n\colon Y_n \in U\} \quad \text{where } Y_n = \{y\colon (n, y) \in P\}.$$

Thus $B \in D$. Choose $y \in B$ and let x be such that $y \in A_x$. Choose $x' > y$ such that $A_{x'} \cap B$ is infinite. This is possible by our assumptions on D and the sequence $\{A_n\colon n \in \omega\}$. Choose $z \in Y_y$ such that $z > x'$ and choose $y' \in A_{x'} \cap B$ such that $y' > z$. This is possible since $A_{x'} \cap B$ is infinite. Finally, choose $z' > y'$ such that $z' \in Y_{y'}$. Then $y < x' < z < y'$ and $y \in A_x$ and $y' \in A_{x'}$. Moreover, $(y, z) \in P$ since $z \in Y_y$ and $(y', z') \in P$ since $z' \in Y_{y'}$. Thus $\{(y, z), (y', z')\} \in [P]^2$ and $f(\{(y, z), (y', z')\}) = 1$, which is the desired contradiction.

Suppose now that U is not a P-point, and let $\{A_n\colon n \in \omega\}$ be a disjoint partition of ω into infinite sets not in U, such that $A_n \cap n = 0$ and, for all $X \in U$, $\{n\colon X \cap A_n$ is infinite$\}$ is infinite. Define f: $[\omega \times \omega]^2 \to 2$ by

$$f(\{(x, z), (x', z')\}) = 1 \quad \text{iff } z \in A_y, z' \in A_{y'}, x < y < z,$$
$$x' < y' < z' \text{ and } y < x' < z < y'.$$

As before, it is clear that f is not constantly one on any three element set. Suppose $P \in D \times U$ and let $B = \{n\colon Y_n \in U\}$, where again, $Y_n = \{m\colon (n, m) \in P\}$. Choose $x \in B$ and choose $y > x$ such that $A_y \cap Y_x$ is infinite. Choose $x' > y$ such that $x' \in B$. Choose $z \in A_y \cap Y_x$ such that $z > x'$. Now choose $z' \in A_{y'} \cap Y_{x'}$ where $y' > z$ and $z' > y'$. Then $f(\{(x, z), (x', z')\}) = 1$ as desired. Hence $D \times U$ is not a 3-arrow ultrafilter and the proof is complete.

REMARK. An easy extension of the proof of Theorem 3.3 shows that if $D \Sigma U_i$ is a k-arrow ultrafilter then D is a k-arrow P-point and $\{i\colon U_i$ is a k-arrow P-point$\} \in D$.

REMARK. Kanamori has pointed out to the authors that an argument similar to that employed in the last part of the proof of Theorem 3.3 establishes the following: If U is a 3-arrow ultrafilter and f is neither constant on a set in U nor finite to one on a set in U then the ultrafilter

$$f^*(U) = \{X \subseteq \omega : f^{-1}(X) \in U\}$$

is a P-point. The importance of this observation is that it shows that 3-arrow ultrafilters are at least as hard to find as P-points (i.e. if there exists a 3-arrow ultrafilter then there exists a P-point).

We now turn our attention to the task of obtaining some positive results concerning products and sums that are k-arrow ultrafilters. Although our final results are rather easily stated, their proofs require a series of somewhat technical lemmas. We begin with two such lemmas which isolate a desirable feature possessed by incompatible P-points.

LEMMA 3.4. *Suppose D is a P-point and D is distinct from each element of the countable collection $\{U_i; i \in \omega\}$ of ultrafilters. Then there exists a set $A \in D$ such that $\omega - A \in \bigcap\{U_i\colon i \in \omega\}$.*

PROOF. For each $i \in \omega$ choose a set $A_i \in D - U_i$. Since D is a P-point there exists a set $A \in D$ such that $A - A_i$ is finite for all $i \in \omega$. It is easy to see that A works.

NOTE. It is easy to see that Lemma 3.4 can fail if D is not a P-point. In fact, the reader familiar with the topology of $\beta\omega - \omega$ will recognize that the conclusion of Lemma 3.4 fails iff D is a limit of the U_i's.

LEMMA 3.5. *Suppose D is a P-point and D is incompatible with each element of the set $\{U_i\colon i \in \omega\}$ of ultrafilters. Let $A \in D$ be such that $B \in \bigcap\{U_i\colon i \in \omega\}$ where $B = \omega - A$. Suppose $\{A_n\colon n \in \omega\}$ and $\{B_n\colon n \in \omega\}$ are (respectively) pairwise disjoint partitions of A and B into nonempty finite sets. Then there exists $A' \subseteq A$ and $B' \subseteq B$ such that $A' \in D$, $B' \in \bigcap\{U_i\colon i \in \omega\}$, and* *such that for all $n \in \omega$, $A' \cap A_n \neq 0$ iff $B' \cap B_n = 0$.*

PROOF. Let

$$D' = \{X \subseteq \omega: \bigcup\{A_n: n \in X\} \in D\},$$

and for each $i \in \omega$ let

$$U_i' = \{X \subseteq \omega: \bigcup\{B_n: n \in X\} \in U_i\}.$$

Then $D' \leqslant_{RK} D$ and $U_i' \leqslant_{RK} U_i$ for each $i \in \omega$. Since D is a P-point, D' is also a P-point, and our assumption guarantees that D' is distinct from each U_i'. Hence, by Lemma 3.4 there exists a set $X \in D'$ such that $\omega - X \in \bigcap\{U_i': i \in \omega\}$. Let $A' = \bigcup\{A_n: n \in X\}$ and let $B' = \bigcup\{B_n: n \in \omega - X\}$. Then $A' \in D$, $B' \in U_i$ and $A' \cap A_n \neq 0$ iff $n \in X$ iff $B' \cap B_n = 0$.

The next few lemmas give some basic results about sums and partitions that will be needed later.

LEMMA 3.6. *Suppose U is an ultrafilter and f: $[\omega]^2 \to k$. If $i < k$ and f is an i-partition for U then $f([X]^2) = \{i\}$ for some infinite set $X \subseteq \omega$.*

PROOF. Let $A(x) = \{y: f(\{x, y\}) = i\}$ and let $A = \{x: A(x) \in U\}$. Since f is an i-partition for U, $A \in U$. Define $\{x_i: i \in \omega\} = X$ inductively, starting with $x_0 \in A$. If $x_0, \ldots, x_{n-1}$ have been defined and are in A then choose $x_n \in A \cap A(x_0) \cap \cdots \cap A(x_{n-1})$. Clearly X works.

LEMMA 3.7. *Suppose $U = D \Sigma U_i$ and f is a 0-partition for U. Define*:

1. $A_{x_0 y_0 x_1} = \{y_1: f(\{(x_0, y_0), (x_1, y_1)\}) = 0\}$.
2. $B_{x_0 y_0} = \{x_1: A_{x_0 y_0 x_1} \in U_{x_1}\}$.
3. $A_{x_0} = \{y_0: B_{x_0 y_0} \in D\}$.
4. $B = \{x_0: A_{x_0} \in U_{x_0}\}$.

Then $B \in D$.

PROOF. We simply unravel the definitions of "0-partition" and "D-sum", i.e.

f is a 0-partition for U
iff $\{(x_0, y_0): \{(x_1, y_1): f(\{(x_0, y_0), (x_1, y_1)\}) = 0\} \in D \Sigma U_i\} \in D \Sigma U_i$
iff $\{x_0: \{y_0: \{(x_1, y_1): f(\{(x_0, y_0), (x_1, y_1)\}) = 0\} \in D \Sigma U_i\} \in U_{x_0}\} \in D$
iff $\{x_0: \{y_0: \{x_1: \{y_1: f(\{(x_0, y_0), (x_1, y_1)\}) = 0\} \in U_{x_1}\} \in D\} \in U_{x_0}\} \in D$
iff $\{x_0: \{y_0: \{x_1: A_{x_0 y_0 x_1} \in U_{x_1}\} \in D\} \in U_{x_0}\} \in D$
iff $\{x_0: \{y_0: B_{x_0 y_0} \in D\} \in U_{x_0}\} \in D$
iff $\{x_0: A_{x_0} \in U_{x_0}\} \in D$
iff $B \in D$.

LEMMA 3.8. *Suppose D is a k-arrow ultrafilter and $U = D \Sigma U_i$. Let f: $[\omega \times \omega]^2 \to 2$ and suppose that $\forall X \in [\omega \times \omega]^k, f([X]^2) \neq \{1\}$. Then*

$$(\exists B \in D)\big(\forall\{x_0, x_1\} \in [B]^2\big)\big(U_{x_0} = U_{x_1} \to E_{\{x_0,x_1\}} \in U_{x_0}\big),$$

where

$$E_{\{x_0,x_1\}} = \big\{y : f\big(\{(x_0, y), (x_1, y)\}\big) = 0\big\}.$$

PROOF. Define g: $[\omega]^2 \to 2$ by $g(\{x_0, x_1\}) = 0$ iff either $U_{x_0} \neq U_{x_1}$ or $E_{\{x_0,x_1\}} \in U_{x_0} = U_{x_1}$. Suppose that there exists some $X \in [\omega]^k$ such that $g([X]^2) = \{1\}$. Then there is some ultrafilter U such that $U_x = U$ for all $x \in X$ such that $\{y: f(\{(x_i, y), (x_j, y)\}) = 1\} \in U$ whenever $\{x_i, x_j\} \in [X]^2$. Choose

$$y \in \bigcap\Big\{\big\{y : f\big(\{(x_i, y), (x_j, y)\}\big) = 1\big\} : \{x_i, x_j\} \in [X]^2\Big\}.$$

Then $f([\{(x_i, y): x_i \in X\}]^2) = \{1\}$ contrary to our assumption. Hence, there exists a set $B \in D$ such that $g([B]^2) = \{0\}$. Clearly B is the desired set.

LEMMA 3.9. *Suppose D is a k-arrow ultrafilter and $U = D \sum U_i$. Let f: $[\omega \times \omega]^2 \to 2$ and suppose that $\forall X \in [\omega \times \omega]^k, f([X]^2) \neq \{1\}$. Then*

$$(\exists B \in D)\big(\forall\{x_0, x_1\} \in [B]^2\big)\big(G_{(x_0,x_1)} \in U_{x_0}\big)$$

where

$$G_{(x_0,x_1)} = \big\{y_0 : A_{x_0 y_0 x_1} \in U_{x_1}\big\}.$$

PROOF. Define g_0: $[\omega]^2 \to 2$ so that if $x_0 < x_1$ then $g_0(\{x_0, x_1\}) = 0$ iff $G_{(x_0,x_1)} \in U_{x_0}$. Define g_1: $[\omega]^2 \to 2$ so that if $x_0 > x_1$ then $g_1(\{x_0, x_1\}) = 0$ iff $G_{(x_0,x_1)} \in U_{x_0}$. It will clearly suffice to show that for each $i < 2$ there exists a set $B_i \in D$ such that $g_i([B_i]^2) = \{0\}$. We consider only the case $i = 0$ since the case $i = 1$ is similar. Suppose for contradiction that $g_0([\{x_0, \ldots, x_{k-1}\}]^2) = \{1\}$. Let

$$A'_{x_0 y_0 x_1} = \big\{y_1 : f\big(\{(x_0, y_0), (x_1, y_1)\}\big) = 1\big\}.$$

For each ordered pair (i, j) with $0 \leqslant i < j \leqslant k - 1$, let

$$G'_{(x_i,x_j)} = \big\{y_0 : A'_{x_i y_0 x_j} \in U_{x_j}\big\}.$$

Since $g_0([\{x_0, \ldots, x_{k-1}\}]^2) = \{1\}$ we have that $G'_{(x_i,x_j)} \in U_{x_i}$ for all i, j with $0 \leqslant i < j \leqslant k - 1$. Choose

$$y_0 \in \bigcap\big\{G'_{(x_0,x_j)} : 0 < j \leqslant k - 1\big\}.$$

Suppose $y_0, \ldots, y_{j-1}$ have been defined. Choose

$$y_j \in \bigcap\big\{G'_{(x_j,x_r)} : j < r \leqslant k - 1\big\} \cap \bigcap\big\{A'_{x_i y_i x_j} : i < j\big\}.$$

Then $f([\{(x_0, y_0), \ldots, (x_{k-1}, y_{k-1})\}]^2) = \{1\}$ contrary to our assumption.

The next lemma is stated without proof since it involves nothing more than a gathering together of the information provided by Lemmas 3.6–3.9. The notation introduced in these lemmas is retained.

LEMMA 3.10. *Suppose D is a k-arrow ultrafilter and $U = D \Sigma U_i$. Let f: $[\omega \times \omega]^2 \to 2$ and suppose that $\forall X \in [\omega \times \omega]^k$, $f([X]^2) \neq \{1\}$. Then there exists a set $K \in D$ such that* (1)–(3) *are satisfied*:

1. *For all $x \in K$, $A_x \in U_x$.*
2. *If $\{x_0, x_1\} \in [K]^2$ and $U_{x_0} = U_{x_1}$, then $E_{\{x_0,x_1\}} \in U_{x_0} = U_{x_1}$.*
3. *If $\{x_0, x_1\} \in [K]^2$, then $G_{(x_0,x_1)} \in U_{x_0}$ and $G_{(x_1,x_0)} \in U_{x_1}$.*

The next lemma is the key result needed for the "positive" theorems that we have been able to obtain concerning partitions of sums and products.

LEMMA 3.11. *Let $\{D, U_0, U_1, \ldots\}$ be a set of k-arrow P-points such that D is incompatible with each U_i and let $U = D \Sigma U_i$. Let f: $[\omega \times \omega]^2 \to 2$ and suppose that $\forall X \in [\omega \times \omega]^k, f([X]^2) \neq \{1\}$. Then there exists a set $P \in U$ and a collection $\{A_n: n \in \omega\}$ of finite subsets of ω satisfying the following*:

1. *For all $n \in \omega$, $\sup(A_n) < \inf(A_{n+1})$.*
2. *If $A = \bigcup\{A_n: n \in \omega\}$ and $B = \omega - A$, then $B \in D$ and $A \in U_x$ for all $x \in B$.*
3. *$P \subseteq B \times A$.*
4. *If $\{(x,y), (x',y')\} \in [P]^2$ and $f(\{(x,y), (x',y')\}) = 1$, then $x \neq x'$ and there is some $n \in \omega$ such that $\{y, y'\} \subseteq A_n$.*

PROOF. We retain the notation of Lemmas 3.7–3.10. Let $K \in D$ satisfy conditions (1)–(3) of Lemma 3.10. By Lemma 3.4 we can clearly assume that $M \in \bigcap\{U_i: i \in \omega\}$ where $M = \omega - K$. We now exploit the fact that D and each U_i is a P-point as follows. Choose $L \in D$ such that $|L - B_{x_0y_0}| < \omega$ for all x_0, y_0 such that $B_{x_0y_0} \in D$. For each $x \in \omega$ choose $N_x \in U_x$ such that $|N_x - A_x|$, $|N_x - A_{x_0y_0x}|$, $|N_x - E_{\{x,x'\}}|$, $|N_x - G_{(x,x')}|$, $|N_x - G_{(x',x)}| < \omega$ whenever $A_x, A_{x_0y_0x}, E_{\{x,x'\}}, G_{(x,x')}, G_{(x',x)} \in U_x$. Finally, let $B' = K \cap L$, $A'_x = M \cap N_x$ and $A' = \omega - B'$.

The next step is to inductively construct a sequence $\{(D_i, C_i): i \in \omega\}$ of pairs of finite sets satisfying the following:

(i) $\sup(C_i) < \inf(C_{i+1})$ and $\sup(D_i) < \inf(D_{i+1})$.

(ii) $A' = \bigcup\{C_n: n \in \omega\}$ and $B' = \bigcup\{D_n: n \in \omega\}$.

(iii) If $y \in C_n$, $x < y$ and $x \in B'$, then $x \in D_0 \cup \cdots \cup D_{n+1}$.

(iv) If $x_0 \in D_0 \cup \cdots \cup D_n$, $y_0 \in C_0 \cup \cdots \cup C_n$ and $B_{x_0y_0} \in D$, then $B' - (D_0 \cup \cdots \cup D_{n+1}) \subseteq B_{x_0y_0}$.

(v) If $\{x_0, x_1\} \in [D_0 \cup \cdots \cup D_n]^2$ then

$$N_{x_0} - (C_0 \cup \cdots \cup C_n) \subseteq A_{x_0} \cap G_{(x_0x_1)} \quad \text{and}$$

$$N_{x_0} - (C_0 \cup \cdots \cup C_n) \subseteq E_{\{x_0,x_1\}} \quad \text{if } U_{x_0} = U_{x_1}.$$

(vi) If $\{x_0, x_1\} \in [D_0 \cup \cdots \cup D_n]^2$, $y_0 \in C_0 \cup \cdots \cup C_{n-1}$ and $A_{x_0y_0x_1} \in U_{x_1}$, then

$$N_{x_1} - (C_0 \cup \cdots \cup C_n) \subseteq A_{x_0y_0x_1}.$$

Let $D_0 = \{b_0\}$ where b_0 is the first element of B'. Choose k large enough so that $N_{b_0} - k \subseteq A_{b_0}$. (Since $b_0 \in B' \subseteq K$ we have that $A_{b_0} \in U_{b_0}$, so $|N_{b_0} - A_{b_0}| < \omega$.) Let $C_0 = \{y \in A': y \leqslant k\}$. This defines (D_0, C_0) so that (i) and the first part of (v) will be satisfied, and the rest are vacuously taken care of. Suppose now that $\{D_0, \ldots, D_n\}$ and $\{C_0, \ldots, C_n\}$ have been defined.

Construction of D_{n+1}. Let

$$r_0 = \sup(\{b \in B': \exists y \in C_0 \cup \cdots \cup C_n\ (b < y)\}).$$

Choose r_1 large enough so that

$$B' - r_1 \subseteq \bigcap \left\{ B_{x_0 y_0}: x_0 \in \bigcup_{i<n} D_i,\ y_0 \in \bigcup_{i<n} C_i,\ B_{x_0 y_0} \in D \right\}.$$

Choose r_2 such that $r_2 > r_0, r_1, \inf(B' - (D_0 \cup \cdots \cup D_n))$. Finally, let

$$D_{n+1} = \{x \in B' - (D_0 \cup \cdots \cup D_n): x \leqslant r_2\}.$$

Construction of C_{n+1}. Notice that if $\{x_0, x_1\} \in [D_0 \cup \cdots \cup D_{n+1}]^2$ then $A_{x_0}, G_{(x_0, x_1)} \in U_{x_0}$ and $E_{\{x_0, x_1\}} \in U_{x_0}$ whenever $U_{x_0} = U_{x_1}$. This is true because $D_0 \cup \cdots \cup D_{n+1} \subseteq B' \subseteq K$. Hence, it is possible to choose s_0 large enough so that the following hold whenever $x_0, x_1 \in D_0 \cup \cdots \cup D_{n+1}$:

(a) $N_{x_0} - s_0 \subseteq E_{\{x_0, x_1\}}$ *if* $x_0 \neq x_1$ and $U_{x_0} = U_{x_1}$.

(b) $N_{x_0} - s_0 \subseteq G_{(x_0, x_1)}$ if $x_0 \neq x_1$.

(c) $N_{x_0} - s_0 \subseteq A_{x_0}$.

(d) $N_{x_1} - s_0 \subseteq A_{x_0 y_0 x_1}$ if $x_0 \neq x_1, y_0 \in C_0 \cup \cdots \cup C_n$ and $A_{x_0 y_0 x_1} \in U_{x_1}$.

Choose $s_1 > s_0, \inf(A' - (C_0 \cup \cdots \cup C_n))$. Finally, let

$$C_{n+1} = \{y \in A' - (C_0 \cup \cdots \cup C_n): y \leqslant s_1\}.$$

This yields the pair (D_{n+1}, C_{n+1}) and completes the construction. The resulting sequence $\{(D_n, C_n): n \in \omega\}$ clearly satisfies conditions (i)–(iv).

At this stage we make use of the fact that D is incompatible with each U_i by repeatedly applying Lemma 3.5 to obtain sets $A \subseteq A'$ and $B \subseteq B'$ such that $B \in D$ and $\{i: A \in U_i\} \in D$ and such that (1′)–(4′) hold:

1′. $A \cap C_n = 0$ or $B \cap D_{n+1} = 0$.

2′. $A \cap C_n = 0$ or $B \cap D_n = 0$.

3′. $A \cap C_n = 0$ or $A \cap C_{n+1} = 0$.

4′. $A \cap C_n = 0$ or $B \cap D_{n-1} = 0$.

This is done as follows. First apply Lemma 3.5 to the partitions $\{C_n: n \in \omega\}$ of A' and $\{D_n': n \in \omega\}$ of $B' - \{b_0\}$ where $D_n' = D_{n+1}$. This yields $B'' \in D$ and $A'' \in \bigcap\{U_i: i \in \omega\}$ such that if $A'' \cap C_n \neq 0$ then $B'' \cap D_{n+1} = 0$. Considering now the partitions $\{D_n: n \in \omega\}$ and $\{C_n: n \in \omega\}$, we obtain from Lemma 3.5 sets $B''' \in D$ and $A''' \in \bigcap\{U_i: i \in \omega\}$ such that if $B''' \cap D_n \neq 0$ then $A''' \cap C_n = 0$. Finally, apply Lemma 3.5 to the partitions $\{D_n: n \in \omega\}$ and $\{C_n': n \in \omega\}$, where $C_n' = C_{n+1}$, to get $B'''' \in D$ and $A'''' \in$

$\bigcap\{U_i: i \in \omega\}$ such that if $B'''' \cap D_n \neq 0$ then $A'''' \cap C_{n+1} = 0$. Let

$$B_0 = \bigcup\{D_n: B' \cap B'' \cap B''' \cap B'''' \cap D_n \neq 0\}.$$

Then $B_0 \in D$. Let

$$A_0 = \bigcup\{C_n: A' \cap A'' \cap A''' \cap A'''' \cap C_n \neq 0 \text{ and } n \text{ even}\}$$

and let

$$A_1 = \bigcup\{C_n: A' \cap A'' \cap A''' \cap A'''' \cap C_n \neq 0 \text{ and } n \text{ odd}\}.$$

Then either $B_0' = \{i: A_0 \in U_i\} \in D$ or $B_1' = \{i: A_1 \in U_i\} \in D$. Without loss of generality assume the former. Finally, let $A = A_0$ and let $B = B_0 \cap B_0'$.

CLAIM. $f(\{(x_0, y_0), (x_1, y_1)\}) = 0$ provided the following are satisfied.

a′. $x_0, x_1 \in B$ and $x_0 \neq x_1$.

b′. $y_0 \in A \cap N_{x_0}$ and $y_1 \in A \cap N_{x_1}$.

c′. $x_0 < y_0$ and $x_1 < y_1$.

d′. Either $y_0 = y_1$ and $U_{x_0} = U_{x_1}$ or $\forall n \in \omega\, (y_0 \notin C_n$ or $y_1 \notin C_n)$.

PROOF OF CLAIM: *Case* 1. $y_0 = y_1$ and $U_{x_0} = U_{x_1}$.

Choose $n \in \omega$ such that $y_0 \in C_n$. By condition (iii) and our assumption that $x_0 < y_0$ and $x_1 < y_1$, we have that $x_0, x_1 \in D_0 \cup \cdots \cup D_{n+1}$. Since $A \cap C_n \neq 0$, we have that $B \cap D_{n+1} = 0$ (by 1′) and $B \cap D_n = 0$ (by 2′). Thus $x_0, x_1 \in D_0 \cup \cdots \cup D_{n-1}$. But then by (v) we have

$$y_0 \in N_{x_0} - (C_0 \cup \cdots \cup C_{n-1}) \subseteq E_{\{x_0, x_1\}}$$

since $U_{x_0} = U_{x_1}$ by (d′). Hence $f(\{(x_0, y_0), (x_1, y_1)\}) = 0$ as desired.

Case 2. $y_0 < y_1$.

Choose $n_0, n_1 \in \omega$ such that $n_0 < n_1$, $y_0 \in C_{n_0}$ and $y_1 \in C_{n_1}$. This is possible by the assumption in (d′). Then, as in Case 1, $x_0 \in D_0 \cup \cdots \cup D_{n_0-1}$ and $x_1 \in D_0 \cup \cdots \cup D_{n_1-1}$. In fact, by (4′) we have $x_0, x_1 \in D_0 \cup \cdots \cup D_{n_1-2}$.

SUBCLAIM. $A_{x_0 y_0 x_1} \in U_{x_1}$.

PROOF OF SUBCLAIM. Suppose first that $x_1 \in D_0 \cup \cdots \cup D_{n_0+1}$. Then, in fact, $x_1 \in D_0 \cup \cdots \cup D_{n_0-2}$ (by (1′) and (4′)). But then

$$y_0 \in N_{x_0} - (C_0 \cup \cdots \cup C_{n_0-2}) \subseteq G_{(x_0, x_1)} \cap G_{(x_1, x_0)}.$$

Hence $A_{x_0 y_0 x_1} \in U_{x_1}$ as desired. Suppose now that $x_1 \notin D_0 \cup \cdots \cup D_{n_0+1}$. Since $x_0 \in B, A_{x_0} \in U_{x_0}$. Thus $x_0 \in D_0 \cup \cdots \cup D_{n_0-1}$ and

$$y_0 \in N_{x_0} - (C_0 \cup \cdots \cup C_{n_0-1}) \subseteq A_{x_0}$$

by (v), so $B_{x_0 y_0} \in D$. Now $x_0 \in D_0 \cup \cdots \cup D_{n_0}$, $y_0 \in C_0 \cup \cdots \cup C_{n_0}$ and $B_{x_0 y_0} \in D$, so we have

$$x_1 \in B - (D_0 \cup \cdots \cup D_{n_0+1}) \subseteq B_{x_0 y_0}$$

by (iv). Hence, $A_{x_0 y_0 x_1} \in U_{x_1}$ by definition of $B_{x_0 y_0}$. This proves the Subclaim.

We can now complete the proof of Case 2. We have: $x_0, x_1 \in D_0 \cup \cdots \cup D_{n_1-2}$; $x_0 \neq x_1$; $y_0 \in C_0 \cup \cdots \cup C_{n_1-2}$ (by (3')); and $A_{x_0y_0x_1} \in U_{x_1}$. Thus, by (vi),

$$y_1 \in N_{x_1} - (C_0 \cup \cdots \cup C_{n_1-1}) \subseteq A_{x_0y_0x_1}$$

so $f(\{(x_0, y_0), (x_1 y_1)\}) = 0$ as desired. This completes the proof of the claim.

In order to complete the proof of Lemma 3.11, we "pare down" the set $B \times A$ to the desired set $P \in U$. First, choose sets $N_x' \in U_x$ for all $x \in B$ such that $f(\{(x, y), (x, y')\}) = 0$ whenever $\{y, y'\} \in [N_x']^2$. This is possible since $U_x \to (U_x, k)^2$. Second, choose sets $N_x'' \in U_x$ so that if $U_{x_1} \neq U_{x_2}$ then $N_{x_1}'' \cap N_{x_2}'' = 0$. Finally, choose $A^x \in U_x$ for each $x \in B$ such that $A^x \subseteq N_x \cap N_x' \cap N_x'' \cap A$ and $A^x \cap x = 0$. Let

$$P = \{(x, y): x \in B \text{ and } y \in A^x\}.$$

Then for every pair $\{(x_0, y_0), (x_1, y_1)\} \in [P]^2$ we have $f(\{(x_0, y_0), (x_1, y_1)\}) = 0$, provided there exists $n, m \in \omega$ such that $n \neq m$, $y_0 \in C_n$ and $y_1 \in C_m$. Clearly then P is the desired set and the proof of Lemma 3.11 is complete.

With Lemma 3.11 now at our disposal, we are ready to derive several "positive" results concerning partitions of sums and products. We begin with the following.

THEOREM 3.12 *Suppose that* $\{D, U_0, U_1, \ldots\}$ *is a set of pairwise incompatible k-arrow P-points, and let* $U = D \sum U_i$. *Then U is a k-arrow ultrafilter.*

PROOF. Choose $B \in D$ and $A^x \in U_x$ for each $x \in B$, and a partition $\{A_n: n \in \omega\}$ of $A = \omega - B$ such that $A^x \subseteq A$ and $f(\{(x, y), (x', y')\}) = 0$ whenever $x = x'$ or $\{x, x'\} \in [B]^2$, $y \in A^x$, $y' \in A^{x'}$ and $\exists n(y \in A_n$ and $y' \notin A_n)$. This is possible, of course, by Lemma 3.11. But now using Lemma 3.5 it is easy to inductively construct sets $A_0^x \subseteq A^x$ for each $x \in B$ such that $A_0^x \in U_x$, and such that if $A_0^x \cap A_n \neq 0$ then $A_0^{x'} \cap A_n = 0$ for all $x' \in B$ with $x' > x$. Then

$$P = \{(x, y): x \in B \text{ and } y \in A_0^x\} \in U$$

and $f([P]^2) = \{0\}$ as desired.

COROLLARY 3.13. *If there exists a countable collection of pairwise nonisomorphic Ramsey ultrafilters, then there exists an arrow Q-point that is not a P-point.*

PROOF. If $\{D, U_0, U_1, \ldots\}$ is a collection of pairwise nonisomorphic Ramsey ultrafilters then, since Ramsey ultrafilters are minimal in the Rudin-Keisler ordering, the elements of this collection are, in fact, pairwise incompatible. Let $U = D \sum U_i$. The partition $\omega \times \omega = \bigcup\{A_n: n \in \omega\}$, where $A_n = \{n\} \times \omega$, shows that U is not a P-point, and Theorem 3.12

guarantees that U is k-arrow for all k with $3 \leqslant k < \omega$. The fact that U is a Q-point follows from a result of Puritz [14].

Corollary 3.13 was one of the primary reasons that we considered sums as opposed to the more restricted class of products. Having obtained this, we now consider products. Let us begin by deriving the obvious corollary of Lemma 3.11.

LEMMA 3.14. *Suppose that D and U are incompatible k-arrow P-points. Let f: $[\omega \times \omega]^2 \to 2$ and suppose that $\forall X \in [\omega \times \omega]^k, f([X]^2) \neq \{1\}$. Then there exist a set $B \in D$, a collection $\{A_x: x \in B\} \subseteq U$ and a partition $\{C_n: n \in \omega\}$ of ω such that the following are satisfied:*

1. *The set $P = \{(x, y): x \in B$ and $y \in A_x\}$ is in $D \times U$.*
2. *For all $n \in \omega$, $\sup(C_n) < \inf(C_{n+1})$.*
3. *If $\{(x, y), (x', y')\} \in [P]^2$ and $f(\{(x, y), (x', y')\}) = 1$, then $x \neq x'$ and $y \neq y'$ and $\exists n \in \omega\, (y, y' \in C_n)$.*

PROOF. Immediate from Lemmas 3.8 and 3.11 and the definition of product.

The next result is the best that we have been able to obtain concerning when a product is a k-arrow ultrafilter.

THEOREM 3.15. *Suppose U is weakly Ramsey, D is a k-arrow P-point, and U and D are incompatible in the Rudin-Keisler ordering. Then $D \times U$ is a k-arrow ultrafilter.*

PROOF. Suppose f, B, A_x, P and $\{C_n: n \in \omega\}$ are as in Lemma 3.14. For each ordered pair $(m, n) \in B \times B$ define $f_{(m,n)}$: $[\omega]^2 \to 2$ so that if $y < y'$ then $f_{(m,n)}(\{y, y'\}) = 0$ iff $f(\{(m, y), (n, y')\}) = 0$.

CLAIM. There is a set $B' \in D$ such that for all $(m, n) \in B' \times B'$ there exists a set $H_{(m,n)} \in U$ such that $f_{(m,n)}([H_{(m,n)}]^2) = \{0\}$.

PROOF OF CLAIM. Define h_0, h_1: $[B]^2 \to 2$ as follows. Let $m, n \in B$ with $m < n$. Then set $h_0(\{m, n\}) = 0$ iff there exists a set $H_{(m,n)}$ as desired, and set $h_1(\{m, n\}) = 0$ iff there exists a set $H_{(n,m)}$ as desired. It suffices to show that there are sets $B_0' \in D$ and $B_1' \in D$ such that $h_i([B_i]^2) = \{0\}$ for $i = 0, 1$. We consider only the case $i = 0$ since the other is analogous. Suppose no such set $B_0' \in D$ exists. Then there exists a set $\{x_0, \ldots, x_{k-1}\} \in [B]^k$ such that $x_0 < \cdots < x_{k-1}$ and $h_0([\{x_0, \ldots, x_{k-1}\}]^2) = \{1\}$. Notice that if $0 \leqslant i < j \leqslant k - 1$ then $f_{(x_i,x_j)}$ is a 0-partition for U. Since U is weakly Ramsey, we can appeal to Theorem 2.4(iii) and obtain a set $\{y_0, \ldots, y_{k-1}\} \in [\omega]^k$ such that $y_0 < \cdots < y_{k-1}$ and $f_{(x_i, x_j)}([\{y_0, \ldots, y_{k-1}\}]^2) = \{1\}$ whenever $0 \leqslant i < j \leqslant k - 1$. But then $f([\{(x_0, y_0), \ldots, (x_{k-1}, y_{k-1})\}]^2) = \{1\}$. This contradiction proves the claim.

To complete the proof of Theorem 3.15, we let B' be as in the claim and choose $A \in U$ such that $|A - H_{(m,n)}| < \omega$ for all $\{m, n\} \in [B']^2$. Enumerate B' in increasing order as $\{b_i: i \in \omega\}$. We define a sequence $\{A'_{b_i}: i \in \omega\} \subseteq U$ as follows. Let $A'_{b_0} = A \cap A_{b_0}$. Suppose $A'_{b_0}, \ldots, A'_{b_n}$ have been defined. For each $i \leqslant n$ choose k_i such that

$$A'_{b_i} - (C_0 \cup \cdots \cup C_{k_i}) \subseteq H_{(b_i,b_{n+1})} \cap H_{(b_{n+1},b_i)}.$$

Choose $d_{n+1} \geqslant \sup(\{k_i: i \leqslant n\})$ such that

$$A - (C_0 \cup \cdots \cup C_{d_{n+1}}) \subseteq \bigcap \{H_{(b_i,b_{n+1})} \cap H_{(b_{n+1},b_i)}: i \leqslant n\}.$$

Let $A'_{b_{n+1}} = (A \cap A_{b_{n+1}}) - (C_0 \cup \cdots \cup C_{d_{n+1}})$. It is now clear that if $P = \{(x, y): x \in B' \text{ and } y \in A'_x\}$ then $P \in D \times U$ and $f([P]^2) = \{0\}$.

COROLLARY 3.16. *If U and D are Ramsey ultrafilters then $U \times D$ is an arrow ultrafilter iff U and D are not isomorphic.*

PROOF. Immediate from Theorems 3.3 and 3.16.

COROLLARY 3.17. *Suppose D and U_i ($i \in \omega$) are Ramsey ultrafilters and D is not isomorphic to any U_i. Let $U = D \sum U_i$. Then U is an arrow ultrafilter.*

PROOF. Since D is Ramsey there exists a set $B \in D$ such that either $U_x \not\cong U_{x'}$ for all $\{x, x'\} \in [B]^2$ or $U_x \cong U_{x'}$ for all $\{x, x'\} \in [B]^2$. In the first case we are done by Theorem 3.12, and in the second case the result follows from Theorem 3.15.

COROLLARY 3.18. *If U and D are weakly Ramsey ultrafilters then $U \times D$ is an arrow ultrafilter iff U and D are incompatible.*

PROOF. Immediate from Theorems 3.3 and 3.16.

4. **Some constructions.** In this section we use a well-known consequence of Martin's axiom [10] to construct several ultrafilters on ω which show that the classes of ultrafilters that we are considering are distinct. This consequence of Martin's axiom that we will employ is the following assertion (denoted "P(c)"):

P(c): If $F \subseteq \mathcal{P}(\omega)$, $|F| < 2^{\aleph_0}$ and finite intersections of elements of F are infinite, then there exists an infinite set $A \subseteq \omega$ such that $A - B$ is finite for all $B \in F$.

It is easy to see that P(c) is an immediate consequence of the continuum hypothesis. A proof that $P(c)$ follows from Martin's axiom $(+2^{\aleph_0} > \aleph_1)$ can be found in [3]. It is worth noting that a consideration of consequences of P(c) + $2^{\aleph_0} > \aleph_1$ long predates the proof of its consistency. In fact, Rothberger [15] proved that P(c) implies the following assertion, P_0(c) (which we will also need). Recall first that if $f, g \in {}^\omega\omega$ (the set of all functions mapping ω to itself) then g is said to *eventually dominate* f (denoted $g \gg f$) iff $\{n \in \omega: g(n) \geqslant$

$f(n)\}$ is cofinite. Also, if g eventually dominates every constant function then g is said to be *eventually arbitrarily large* (abbreviated by "g is e.a.l.").

$P_0(c)$. If $G \subseteq {}^\omega\omega$, $|G| < 2^{\aleph_0}$, and every element $g \in G$ is e.a.l., then there exists $f \in {}^\omega\omega$ such that f is e.a.l. and $g \gg f$ for every $g \in G$.

It can be shown that $P_0(c)$ is equivalent to the assertion that every family $G \subseteq {}^\omega\omega$ of size less than $2^{\aleph_0}$ can be eventually dominated. (i.e. $\exists f \forall g \in G$, $g \ll f$).

Assumptions such as CH, MA or P(c) are useful in constructing ultrafilters in that they often allow inductive constructions that involve handling $2^{\aleph_0}$ conditions compatibly (e.g. to construct a Ramsey ultrafilter one enumerates the set $\{f_\alpha: \alpha < 2^{\aleph_0}\}$ of all partitions f: $[\omega]^2 \to 2$ and at stage α insures that f_α will not be a counterexample to the Ramseyness of the ultrafilter being constructed). When constructing ultrafilters that have some "positive property" (e.g. weak-Ramseyness) and also some "negative property" (e.g. non-Ramseyness), one is faced with two problems:

1. The preservation of an inductive hypothesis based on the negative property.

2. The combinatorial result needed at stage α to obtain the positive property.

There is a uniform way to handle the first problem using P(c), and in several different cases the arguments involved in dispensing with the second problem are similar to one another. In view of this, we have chosen to begin by developing some machinery in the form of a series of definitions and lemmas that will allow us to then derive the existence of several types of ultrafilters by simply appealing to the lemmas and establishing an appropriate combinatorial result (without ever mentioning P(c) or ultrafilters). In fact, it turns out in most cases that the combinatorial result needed is a theorem of finite combinatorics. It should also be mentioned that the machinery developed here has been employed in work as yet unpublished to simplify the construction of several kinds of ultrafilters involving classes other than those considered in the present paper.

Definition 4.1. If $\mathfrak{D} = \{D_n: n \in \omega\}$ is a partition of ω into pairwise disjoint sets then a $\mathfrak{D}$*-norm* N is an integer valued function satisfying the following:

1. $X \in$ Domain(N) iff $X \subseteq D_n$ for some $n \in \omega$.
2. $N(\emptyset) = 0$.
3. If $X, Y \subseteq D_n$ then $N(X \cup Y) \leq N(X) + N(Y)$.
4. If $X \subseteq Y \subseteq D_n$ then $N(X) \leq N(Y)$.

If N is a $\mathfrak{D}$-norm and $X \subseteq \omega$, then we define $g_X^N \in {}^\omega\omega$ and $\mathcal{G}_N \subseteq \mathcal{P}(\omega)$ by:

$$g_X^N(n) = \sup\{N(X \cap D_i): i \leqslant n\},$$

$$\mathcal{G}_N = \{X \subseteq \omega: g_X^N \text{ is eventually arbitrarily large}\}.$$

The $\mathfrak{D}$-norm N is said to be nontrivial iff $\mathcal{G}_N \neq 0$. If $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$ is a collection of propositions pertaining to subsets of ω, then we say that the $\mathfrak{D}$-norm N *handles* Φ iff the following is satisfied:

$$\forall \alpha < 2^{\aleph_0} \forall X \in \mathcal{G}_N \exists Y \subseteq X [Y \in \mathcal{G}_N \text{ and } \phi_\alpha(Y) \text{ holds}].$$

EXAMPLE 4.2. Let $\mathfrak{D} = \{D_n: n \in \omega\}$ be a partition of ω into pairwise disjoint sets where $|D_n| = n + 1$. Let the $\mathfrak{D}$-norm N be defined as follows.

$$\text{If } s \subseteq D_n \text{ then } N(s) = |s|.$$

Then $\mathcal{G}_N = \{X \subseteq \omega: \forall n \exists m(|X \cap D_m| \geqslant n)\}$. Hence, N is nontrivial. Let $\{P_\alpha: \alpha < 2^{\aleph_0}\}$ be an enumeration of all partitions of ω where $P_\alpha = \{A_n^\alpha: n \in \omega\}$. Let $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$ where $\phi_\alpha(X)$ is the following assertion:

$$\phi_\alpha(X): \exists n \in \omega\, (X \subseteq A_n^\alpha) \text{ or } \forall n \in \omega\, (X \cap A_n^\alpha \text{ is finite}).$$

It is easy to check that N handles Φ.

REMARK. It follows from 4.4 below that if $\mathcal{G}_N$ is as in Example 4.2 and if $F \subseteq \mathcal{G}_N$, $|F| < 2^{\aleph_0}$, and F is closed under finite intersections, then there exists a set $X \in \mathcal{G}_N$ such that $X - B$ is finite for all $B \in F$. This remark, together with Example 4.2, allows one to construct (using P(c)) an ultrafilter U that is a P-point but not a Q-point. The P-pointness of U is assured by the fact that N handles Φ and the non-Q-pointness is assured by making $U \subseteq \mathcal{G}_N$.

LEMMA 4.3. *Suppose* $\mathfrak{D} = \{D_n: n \in \omega\}$ *is a partition of* ω *and let* N *be a* $\mathfrak{D}$*-norm. Then*

(i) *If* $x \in \mathcal{G}_N$ *and* $X \subseteq Y$, *then* $Y \in \mathcal{G}_N$.

(ii) *If* X *is finite, then* $X \notin \mathcal{G}_N$.

(iii) *If* $X \notin \mathcal{G}_N$ *and* $Y \notin \mathcal{G}_N$, *then* $X \cup Y \notin \mathcal{G}_N$.

Thus, $\mathcal{P}(\omega) - \mathcal{G}_N$ *is an ideal on* ω *containing all finite sets and it is proper iff* N *is nontrivial.*

PROOF. Immediate from Definition 4.1.

LEMMA 4.4. *Assume* P(c). *Let* $\mathfrak{D} = \{D_n: n \in \omega\}$ *be a partition of* ω *into finite sets and let* N *be a nontrivial* $\mathfrak{D}$*-norm. Suppose* $F \subseteq \mathcal{G}_N$, $|F| < 2^{\aleph_0}$ *and* F *is closed under finite intersections. Then there exists a set* $A \in \mathcal{G}_N$ *such that, for all* $B \in F$, $A - B$ *is finite.*

PROOF. Since $F \subseteq \mathcal{G}_N$, the collection $G = \{g_B^N: B \in F\}$ satisfies the hypotheses of P_0(c). Thus, there is a function $f \in {}^\omega\omega$ such that f is eventually arbitrarily large but f is eventually dominated by g_B^N for every $B \in F$. For every $B \in F$, let Y_B be defined as follows:

$$Y_B = \{s: \exists n\,(s \subseteq B \cap D_n \text{ and } N(s) \geq f(n))\}.$$

Notice first that if $B \in F$ then Y_B is infinite (since $g_B^N \gg f$). Also, if $B_0, \ldots, B_{k-1} \in F$ and $B = B_0 \cap \cdots \cap B_{k-1}$, then $Y_B = Y_{B_1} \cap \cdots \cap Y_{B_{k-1}}$. Thus $\{Y_B: B \in F\}$ is a collection of fewer than $2^{\aleph_0}$ infinite subsets of a countable set and is closed under intersections. Thus, by P(c), there exists an infinite set Y such that $Y - Y_B$ is finite for every $B \in F$. Let $A = \bigcup Y$. Notice first that $A - B$ is finite whenever $B \in F$. To see this, let m be such that $\bigcup(Y - Y_B) \subseteq D_0 \cup \cdots \cup D_{m-1}$. Now, if $x \in A - (D_0 \cup \cdots \cup D_{m-1})$, then $x \in s$ for some $s \in Y$, so $s \subseteq D_k$ for some $k \geq m$, and thus $s \in Y_B$. Hence $s \subseteq B \cap D_k$, so $x \in B$. This shows $A - B$ is finite. Finally, we claim that $A \in \mathcal{G}_N$. This follows since $Y \cap Y_B$ is infinite (for all, hence some, $B \in F$) and f is e.a.l.

LEMMA 4.5. *Assume* P(c). *Let* $\mathfrak{D} = \{D_n: n \in \omega\}$ *be a partition of* ω *into finite sets and let* N *be a nontrivial* $\mathfrak{D}$*-norm. Suppose* N *handles* $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$. *Then there exists an ultrafilter* U *on* ω *satisfying the following*:

1. $\forall \alpha < 2^{\aleph_0} \exists X \in U$ *(*$\phi_\alpha(X)$ *holds).*
2. $U \subseteq \mathcal{G}_N$.

PROOF. Suppse $\alpha < 2^{\aleph_0}$ and $\{X_\beta: \beta < \alpha\}$ has been constructed so that $X_{\beta_1} \cap \cdots \cap X_{\beta_k} \in \mathcal{G}_N$ whenever $\beta_1 < \cdots < \beta_k < \alpha$. Let $F = \{X_{\beta_1} \cap \cdots \cap X_{\beta_k}: \beta_1 < \cdots < \beta_k < \alpha\}$. By Lemma 4.4 there exists a set $A \in \mathcal{G}_N$ such that $A - B$ is finite for all $B \in F$. Since N handles Φ there exists a set $X_\alpha \subseteq A$ such that $X_\alpha \in \mathcal{G}_N$ and $\phi_\alpha(X_\alpha)$ holds. If $B \in F$ then $X_\alpha = (X_\alpha \cap B) \cup (X_\alpha - B)$, so $X_\alpha \cap B \in \mathcal{G}_N$ by Lemma 4.3(ii) and (iii). Lemma 4.3 now guarantees that $\{X_\alpha: \alpha < 2^{\aleph_0}\} \cup \{Y \subseteq \omega: \omega - Y \notin \mathcal{G}_N\}$ has the finite intersection property and so can be extended to the desired ultrafilter U.

LEMMA 4.6. *Let* $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$ *be the "P-point conditions" given in Example* 4.2. *Let* $\mathfrak{D} = \{D_n: n \in \omega\}$ *be a partition of* ω *into finite sets and let* N *be a nontrivial* $\mathfrak{D}$*-norm. Then* N *handles* Φ.

PROOF. Suppose $X \in \mathcal{G}_N$ and $P_\alpha = \{A_n^\alpha: n \in \omega\}$. For each $n \in \omega$ let $A_n = X \cap A_n^\alpha$ and assume that for all $n \in \omega$, $A_n \notin \mathcal{G}_N$. Then by Lemma 4.3, $A_0 \cup \cdots \cup A_n \notin \mathcal{G}_N$ and $X - (A_0 \cup \cdots \cup A_n) \in \mathcal{G}_N$. For each $n \in \omega$ choose $k_n \in \omega$ such that $N(D_{k_n} \cap X - (A_0 \cup \cdots \cup A_n)) > n$ and let $s_n = D_{k_n} \cap X - (A_0 \cup \cdots \cup A_n)$. Let $X' = \bigcup\{s_n: n \in \omega\}$. Then $X' \subseteq X$, $X' \in \mathcal{G}_N$ and $|X' \cap A_n| < \omega$ for all $n \in \omega$.

LEMMA 4.7. *Let* $\mathfrak{D} = \{D_n: n \in \omega\}$ *be a partition of* ω *into finite sets and let* N *be a nontrivial* $\mathfrak{D}$*-norm. Suppose* $X \in \mathcal{G}_N$ *and* $f: [X]^2 \to k$ *where* $k < \omega$. *Then there exist* $Y \subseteq X$ *and* $i < k$ *such that* $Y \in \mathcal{G}_N$ *and* $f(\{x, y\}) = i$ *whenever* $\{x, y\} \in [Y]^2$, $x \in D_n$, $y \in D_m$ *and* $n \neq m$.

REMARK. The following proof of Lemma 4.7 assumes P(c). This can be avoided, but at the expense of some complications.

PROOF. By Lemmas 4.5 and 4.6 there exists an ultrafilter U on ω such that $X \in U$, $U \subseteq \mathcal{G}_N$ and U is a P-point. By Theorem 2.3(iii) there exists a set $Y' \in U$ such that Y' is eventually homogeneous for f. Then $X \cap Y' \in U$ so $X \cap Y' \in \mathcal{G}_N$. Construct $\{s_n: n \in \omega\} \subseteq X \cap Y'$ inductively as follows. If $s_0, \ldots, s_n$ have been constructed then choose k large enough so that f is homogeneous on all pairs $\{x, y\}$ with $x \in s_0 \cup \cdots \cup s_n$ and $y \in X \cap Y' - (D_0 \cup \cdots \cup D_k)$. Choose $k' > k$ such that $N(X \cap Y' \cap D_{k'}) \geqslant n + 1$ and let $s_{n+1} = X \cap Y' \cap D_{k'}$. Finally, let $Y = \bigcup\{s_n: n \in \omega\}$. Clearly Y is the desired set.

Having developed this machinery, we now turn to the task of constructing the ultrafilters showing that the classes we have been considering are distinct. The kinds of ultrafilters that we want are all non-Ramsey and there are three distinct types of non-Ramsey ultrafilters:

1. those that are P-points but not Q-points;
2. those that are Q-points but not P-points;
3. those that are neither P-points nor Q-points.

Our constructions of type 1 will depend on the machinery developed in this section. The constructions of type 2 will depend upon considering a suitable sum. Constructions of type 3 are similar to type 2 but can sometimes be carried out without any extra set theoretic assumptions.

We begin with some of type 1, the first two of which are known but are included for completeness since they follow immediately from the machinery developed.

THEOREM 4.8 (BOOTH [3] WITH MA). *Assume* P(c). *Then there exists a P-point that is not a Q-point* (*and thus not Ramsey*).

PROOF. Let $\mathfrak{D} = \{D_n: n \in \omega\}$ be a partition of ω into finite sets where $|D_n| = n + 1$. Define a $\mathfrak{D}$-norm N as follows. If $s \subseteq D_n$ then $N(s) = |s|$. It is trivial to check that N is a nontrivial $\mathfrak{D}$-norm, so the theorem follows from Lemmas 4.5 and 4.6.

THEOREM 4.9 (BLASS [2] WITH CH). *Assume* P(c). *Then there exists a weakly Ramsey ultrafilter that is not a Q-point* (*and thus not Ramsey*).

PROOF. Let $\mathfrak{D}$ and N be as in the proof of Theorem 4.8. Let $\{f_\alpha: \alpha < 2^{\aleph_0}\}$ enumerate all partitions $f: [\omega]^2 \to 3$. Let $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$, where $\phi_\alpha(X)$ is the assertion: $|f_\alpha([X]^2)| \leqslant 2$. By Lemmas 4.5 and 4.6 it suffices to show that N handles Φ. We can suppose then, by Lemma 4.7, that $X \in \mathcal{G}_N$ and $f(\{x, y\}) = i$ whenever $\{x, y\} \in [Y]^2$, $x \in D_n$, $y \in D_m$ and $n \neq m$. Since $Y \in \mathcal{G}_N$ we can use the finite version of Ramsey's theorem to choose distinct

k_n's for $n \in \omega$ such that $|Y \cap D_{k_n}| \to (n)^2_3$. Thus, there exist $s_n \in [Y \cap D_{k_n}]^n$ and $i_n < 2$ such that $f([s_n]^2) = \{i_n\}$. Since $S = \bigcup\{s_n: n \in \omega\} \in \mathcal{G}_N$ we have (by Lemma 4.3) that $S_j \in \mathcal{G}_N$ for some $j < n$, where $S_j = \bigcup\{s_n: i_n = j\}$. Thus $S_j \subseteq X$ and $S_j \in \mathcal{G}_N$, and $f_\alpha([S_j]^2) \subseteq \{i, j\}$, so $\phi_\alpha(S_j)$ holds. This shows that N handles ϕ and completes the proof.

THEOREM 4.10. *Assume* P(c). *Then there exists an arrow P-point that is not weakly Ramsey.*

PROOF. Let $\mathfrak{D} = \{D_n: n \in \omega\}$ be a disjoint partition of ω such that $|D_n| = 2^{[n/2]}$. Define a $\mathfrak{D}$-norm N as follows: If $s \subseteq D_n$ then $N(s) = m$ iff $n^{m-1} \leqslant |s| < n^m$ (and $N(\emptyset) = 0$).

It is easy to check that N is a nontrivial $\mathfrak{D}$-norm. In fact,

$$\mathcal{G}_N = \{X \subseteq \omega: \forall m \exists n |X \cap D_n| \geqslant n^m\}$$

and $\omega \in \mathcal{G}_N$ since $\lim_{n\to\infty} n/2\log(n) = \infty$. Let $\{f_\alpha: \alpha < 2^{\aleph_0}\}$ enumerate all partitions f: $[\omega]^2 \to 2$ having the property that $\exists k \in \omega, f([X]^2) \neq \{1\}$ for any $X \in [\omega]^k$. Let $\Phi = \{\phi_\alpha \wedge \psi_\alpha: \alpha < 2^{\aleph_0}\}$ where ϕ_α is the "P-point condition" from Example 4.2 and $\psi_\alpha(X)$ is the assertion: $f_\alpha([X]^2) = \{0\}$. By Lemma 4.6, it suffices to show that N handles $\Psi = \{\psi_\alpha: \alpha < 2^{\aleph_0}\}$ in order to conclude that N handles Φ.

Suppose then that $X \in \mathcal{G}_N$ and f: $[\omega]^2 \to 2$ is such that $f([X]^2) \neq \{1\}$ for any $X \in [\omega]^k$. By Lemma 4.7 there exist $Y \subseteq X$ and $i < 2$ such that $Y \in \mathcal{G}_N$ and $f(\{x, y\}) = i$ whenever $\{x, y\} \in [Y]^2$, $x \in D_n$, $y \in D_m$ and $n \neq m$. Clearly we must have $i = 0$. Hence, to show that N handles Φ it suffices to show that for every $m \in \omega$ there exist $n \in \omega$ and $s \subseteq Y \cap D_n$ such that $|s| \geqslant n^m$ and $f([s]^2) = \{0\}$. Given m choose $n \in \omega$ such that $|Y \cap D_n| \geqslant n^{m\cdot k}$. The following result from finite combinatorics is easy to establish by induction on k:

$$n^k \to (n, k+1)^2.$$

Hence $(n^m)^k \to (n^m, k+1)^2$. Since $|Y \cap D_n| \geqslant n^{mk}$ and $f([X]^2) \neq \{1\}$ for any $X \in [\omega]^k$, we have that there is some set $s \subseteq Y \cap D_n$ such that $|s| \geqslant n^m$ and $f([s]^2) = \{0\}$. This shows that N handles $\{\psi_\alpha: \alpha < 2^{\aleph_0}\}$ and so N handles Φ.

Lemma 4.6 now guarantees the existence of an ultrafilter $U \subseteq \mathcal{G}_N$ such that for every $\alpha < 2^{\aleph_0}$ there is a set $X \in U$ such that $\phi_\alpha(X)$ and $\psi_\alpha(X)$ hold. This guarantees that U is an arrow P-point.

It remains only to show that U is not weakly Ramsey. A classical result of Erdös shows that $2^{n/2} \not\to (n)^2_2$, so for each $n \in \omega$ there exists a function f_n: $[D_n]^2 \to 2$, such that if $s \subseteq D_n$ and f_n is constant on $[s]^2$, then $|s| \leqslant n$. Define g: $[\omega]^2 \to 3$ by

$$g(\{x,y\}) = \begin{cases} f_n(\{x,y\}) & \text{if } \{x,y\} \in [D_n]^2, \\ 2 & \text{if } x \in D_n \text{ and } y \in D_m \text{ and } n \neq m. \end{cases}$$

If $|g([X]^2)| < 2$ and X meets more than one D_n, then $|X \cap D_n| < n$, so $X \notin \mathcal{G}_N$. Since $U \subseteq \mathcal{G}_N$ this forces $X \notin U$. If $X \subseteq D_n$ for some n, then we also have $X \notin U$. Hence g shows that U is not weakly Ramsey.

In a preliminary version of this paper the authors proved that one could construct (using P(c)) a P-point that is not a 3-arrow ultrafilter. The question as to whether one could construct, say, a 3-arrow ultrafilter that is not 4-arrow was left open. The difficulty encountered in such a construction rests with the complexity of the finite combinatorics involved. It was Fred Galvin who settled this question by pointing out that the appropriate result needed is a recent (and quite deep) theorem of Nešetřil and Rödl [13]. If G is a finite graph then we let $V(G)$ and $E(G)$ denote the sets of vertices of G and edges of G, respectively, and $\mathrm{cl}(G)$ denotes the clique number of G (i.e. the largest integer k such that G contains a complete subgraph of size k). Let $\mathrm{Cl}(k)$ denote the class of all finite graphs G such that $\mathrm{cl}(G) = k$. The Nešetřil-Rödl theorem can now be stated as follows:

For every $k > 0$ there exists a function NR_k: $\mathrm{Cl}(k) \to \mathrm{Cl}(k)$ having the property that if $NR_k(G) = H$ and if f: $[V(H)]^2 \to 2$, then there exists an (induced) subgraph G' of H such that G' is isomorphic to G and

$$|f(E(G'))| = 1 = \left|f\left([V(G')]^2 - E(G')\right)\right|$$

(i.e. f is constant on all pairs of adjacent vertices from G' and f is constant on all pairs of independent vertices from G').

Using the Nešetřil-Rödl theorem, we can now combine our result on P-points that are not 3-arrow with Galvin's theorem in the following:

THEOREM 4.11. *Assume* P(c). *If $2 \leq k < \omega$ then there exists a P-point U such that $U \to (U, k)^2$ but $U \not\to (U, k+1)^2$.*

PROOF. Fix k with $2 \leq k < \omega$. Let C_k denote the complete graph on k points if $k \geq 3$ and let C_k denote any 3 point graph with one edge if $k = 2$. Let NR_k^n denote the n-fold composition of the function NR_k from the Nešetřil-Rödl theorem (with $NR_k^0 =$ identity). Let $\mathfrak{D} = \{D_n: n \in \omega\}$ be a disjoint partition of ω such that $|D_n| = |V(NR_k^n(C_k))|$. We will regard D_n as being equipped with a graph isomorphic to $NR_k^n(C_k)$.

Define the integer valued function N as follows:

If $s \subseteq D_n$ then $N(s) = p$ iff $s = 0$ and $p = 0$ or $s \neq 0$ and $p - 1 = \inf\{m \in \omega$: s contains no isomorphic copy of $NR_k^m(C_k)\}$.

We claim that N is a $\mathfrak{D}$-norm. To see this, suppose X, $Y \subseteq D_n$ with $N(X) = k_1$ and $N(Y) = k_2$. Assume, for contradiction, that $N(X \cup Y) \geq$

$k_1 + k_2 + 1$. Then clearly $k_1 > 0$ and $k_2 > 0$ (since $N(s) = 0$ iff $s = 0$). Choose $H \subseteq X \cup Y$ such that H is isomorphic to $NR_k^{k_1+k_2-1}(C_k)$. Define f: $[V(H)]^2 \to 2$ by $f(\{a, b\}) = 0$, iff $\{a, b\} \subseteq X$ or $\{a, b\} \subseteq Y$. Then there exists an (induced) subgraph G' of H such that G' is isomorphic to $NR_k^{(k_1-1)+(k_2-1)}(C_k)$, and either $|G'| \leq 2$ or else $G' \subseteq X$ or $G' \subseteq Y$. If $|G'| < 2$ then we must have $k_1 = 1$, $k_2 = 1$ and $k = 2$, in which case we clearly have $N(X \cup Y) \leq 2$. If, say, $G' \subseteq X$ then X contains an isomorphic copy of $NR_k^{k_1-1}(C_k)$, so $N(X) \geq k_1 + 1$. Similarly, if $G' \subseteq Y$ then $N(Y) \geq k_2 + 1$ and we have the desired contradiction.

Notice also that the $\mathfrak{D}$-norm N is nontrivial since $\omega \in \mathcal{G}_N$. Moreover, if U is any ultrafilter on ω such that $U \subseteq \mathcal{G}_N$, then $U \nrightarrow (U, k + 1)^2$, as can be seen by considering the function f: $[\omega]^2 \to 2$ where $f(\{x, y\}) = 1$ iff there is some n such that x and y are adjacent in the (isomorphic copy of the) Nešetřil-Rödl graph $NR_k^n(C_k)$ on D_n.

Let $\{f_\alpha: \alpha < 2^{\aleph_0}\}$ enumerate all partitions f: $[\omega]^2 \to 2$ with the property that if $f([X]^2) = \{1\}$ then $|X| < k$. Let $\phi_\alpha(x)$ be the assertion that $f_\alpha([X]^2) = \{0\}$. We claim that the $\mathfrak{D}$-norm N handles $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$. Suppose then that $\alpha < 2^{\aleph_0}$ and $X \in \mathcal{G}_N$. By Lemma 4.7 there exists a set $Y \subseteq X$ such that $Y \in \mathcal{G}_N$ and such that $f_\alpha(\{x, y\}) = 0$ whenever $\{x, y\} \in [Y]^2$, $x \in D_n$, $y \in D_m$ and $n \neq m$. For $m \in \omega$ let $\mathfrak{N}_m = \{s: N(s) \geq m\}$. Then the Nešetřil-Rödl theorem shows that for sufficiently large m we have $\mathfrak{N}_{m+1} \to (\mathfrak{N}_m, k)^2$. To see this, suppose $s \in \mathfrak{N}_{m+1}$ and f: $[s]^2 \to 2$ is such that $|X| < k$ whenever $f([X]^2) = \{1\}$. Then there exists $t \subseteq s$ such that $t \in \mathfrak{N}_m$, and there exists n_a and n_i such that $f(\{x, y\}) = n_a$ whenever $\{x, y\} \in [t]^2$ and x and y are adjacent, and $f(\{x, y\}) = n_i$ whenever $\{x, y\} \in [t]^2$ and x and y are independent. Since the clique number of t is k we must have $n_a = 0$. Moreover, if m is large then $|t|$ is large enough so that Ramsey's theorem guarantees that we must also have $n_i = 0$. Hence, we obtain a set $Y' \subseteq Y$ such that $Y' \in \mathcal{G}_N$ and $\phi_\alpha(Y')$. Thus N handles $\Phi = \{\phi_\alpha: \alpha < 2^{\aleph_0}\}$. Lemma 4.5 now guarantees the existence of the desired ultrafilter.

COROLLARY 4.12. *Assume* P(c). *Then there exists a P-point that is not a 3-arrow ultrafilter.*

PROOF. This is simply the case $k = 2$ of Theorem 4.11.

Theorems 4.8–4.11 give the necessary types of non-Q-points needed to show the classes are distinct. We now turn to the construction of a couple of non-P-points that are needed. As indicated previously, special types of non-P-points can often be found by considering suitable sums. Of course, the existence of the types of ultrafilters needed to form an appropriate sum may depend on some extra set theoretic assumption like P(c). Nevertheless, there is an advantage to avoiding an outright inductive construction. Namely, proofs involving products and sums often generalize to yield interesting classes of κ-complete ultrafilters on a measurable cardinal κ. This will be pursued somewhat further in the next section.

THEOREM 4.13. *Suppose $\{U_i: i \in \omega\}$ is a set of pairwise nonisomorphic Ramsey ultrafilters. Let $2 \leqslant k < \omega$, let D be a k-arrow P-point that is not $(k+1)$-arrow and assume that D is incompatible with each U_i. Let $U = D \sum U_i$. Then U is a k-arrow Q-point that is not $(k+1)$-arrow.*

PROOF. The fact that U is a k-arrow ultrafilter follows from Theorem 3.12, and U cannot be $(k+1)$-arrow since the $(k+1)$-arrow ultrafilters are closed downward in the Rudin-Keisler ordering and $D \leqslant_{RK} U$. The fact that U is a Q-point is a result of Puritz [14]. (This also follows quite easily from Lemmas 3.4 and 3.5.)

COROLLARY 4.14. *Assume* P(c). *If $2 \leqslant k < \omega$ then there exists a Q-point U such that $U \rightarrow (U, k)^2$ but $U \not\rightarrow (U, k+1)^2$.*

COROLLARY 4.15. *Assume* P(c). *Then there exists a Q-point that is not a* 3-*arrow ultrafilter.*

PROOF. This is simply the case $k = 2$ of Corollary 4.14.

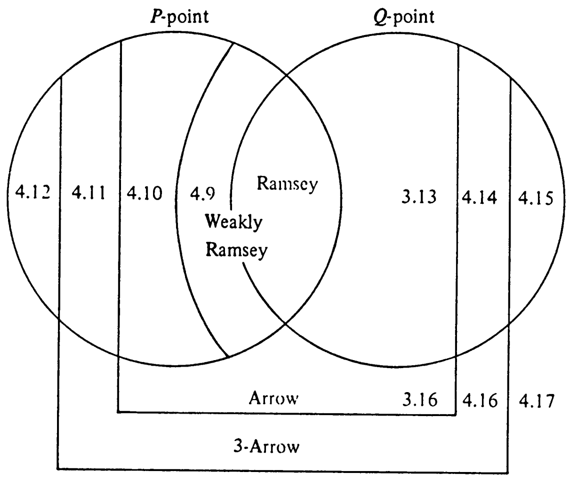
THEOREM 4.16. *Assume* P(c). *If $3 \leqslant k < \omega$ then there exists a k-arrow ultrafilter that is not $(k+1)$-arrow, not a P-point and not a Q-point.*

PROOF. If U is weakly Ramsey and D is a k-arrow P-point that is not $(k+1)$-arrow, and if U and D are incompatible in the Rudin-Keisler ordering, then $D \times U$ is the desired ultrafilter, as can be seen from Theorem 3.15 and the fact that the $(k+1)$-arrow ultrafilters are closed downward in the Rudin-Keisler ordering.

THEOREM 4.17. *There exists an ultrafilter U that is not a P-point, not a Q-point and not a* 3-*arrow ultrafilter.*

PROOF. Let $U = V \times V$ where V is any ultrafilter on ω. Then it is well known that U is neither a P-point nor a Q-point. The fact that U is not a 3-arrow ultrafilter follows from Theorem 3.3.

The following Venn diagram illustrates the relative positions (assuming P(c)) of the classes of ultrafilters that we have been considering. The numbers indicate the theorems showing that the respective regions are nonempty.



5. **Extensions to measurable cardinals.** In this section we briefly consider some possible extensions of the results in §§1–4 to the case where U is a κ-complete ultrafilter on the uncountable measurable cardinal κ. For brevity, we will refer to such an ultrafilter U as a *κ-ultrafilter*. The generalizations of the definitions of P-point, Q-point, etc. to this situation are the natural ones and can be found in either [6] or [7]. We being by collecting together some known results in the following.

THEOREM 5.1. (i) (ROWBOTTOM [16]). *A κ-ultrafilter U is Ramsey iff U is isomorphic to a normal ultrafilter on κ.*

(ii) (KETONEN [7]). *A κ-ultrafilter U is a Q-point iff U is isomorphic to a κ-ultrafilter D that extends the closed unbounded filter on κ.*

(iii) *Assume that κ is a measurable cardinal and a limit of measurable cardinals. Let U be a normal ultrafilter on κ and for each measurable cardinal $\mu < \kappa$ let D_μ be a normal ultrafilter on μ. For each $\alpha < \kappa$ let $m(\alpha)$ be the least measurable cardinal greater than α. Define the κ-ultrafilter D on κ by*

$$X \in D \text{ iff } \{\alpha < \kappa : X \cap m(\alpha) \in D_{m(a)}\} \in U.$$

Then: (a) (KETONEN [7]). *D is a P-point but D is not a Q-point.*

(b) (KANAMORI [6]). *D is a weakly Ramsey κ-ultrafilter.*

The next theorem shows that the mere assumption of the existence of an uncountable measurable cardinal is not enough to allow us to prove much about these classes of ultrafilters.

THEOREM 5.2. *Suppose U is a normal ultrafilter on κ and $V = L[U]$. Then the classes of Ramsey ultrafilters, P-points, Q-points, weakly-Ramsey ultrafilters and 3-arrow ultrafilters all coincide.*

PROOF. If D is isomorphic to the unique normal ultrafilter U on κ then D belongs to all the classes. If D is not isomorphic to U then a result of Kunen [9] shows that D is isomorphic to U^n for some n such that $2 \leq n < \omega$. A product is never a Q-point or a P-point (and therefore neither Ramsey nor weakly-Ramsey). Moreover, the argument given in the proof of Theorem 3.3 extends easily to show that U^n is not a 3-arrow ultrafilter.

The results in §2 extend easily to yield the following where U is a κ-ultrafilter on the uncountable measurable cardinal κ.

THEOREM 5.3. (i) *If $U \to (U, \omega)^2$ then U is a P-point.*
(ii) *If $U \to (U, \kappa)^2$ then U is a Q-point.*
(iii) *U is Ramsey iff $U \to (U, \kappa)^2$ iff $U \to (U, 4)^3$.*
(iv) *If U is weakly Ramsey then $U \to (U, \lambda)^2$ for all $\lambda < \kappa$.*

As a sample of an extension of the results in §3, we mention the following.

THEOREM 5.4. *Suppose D and $\{U_\alpha: \alpha < \kappa\}$ are normal ultrafilters on κ and D is distinct from each U_α. Then $D \Sigma U_\alpha$ is an arrow ultrafilter (i.e. $D \Sigma U_\alpha \to (D \Sigma U_\alpha, n)^2$ for all $n \in \omega$). In particular, the product of distinct normal ultrafilters on κ is an arrow ultrafilter.*

THEOREM 5.2 showed that the mere assumption that κ is an uncountable measurable cardinal is not sufficient to yield any results analogous to those of §4 concerning P-points, Q-points and arrow ultrafilters. However, the techniques of this section do yield the following.

THEOREM 5.5. *Suppose that κ is the κth measurable cardinal and that κ carries κ normal ultrafilters. Then there are κ-ultrafilters U_1, U_2 and U_3 on κ having the following properties.*
1. *U_1 is an arrow P-point but not a Q-point.*
2. *U_2 is an arrow Q-point that is not a P-point.*
3. *U_3 is a Q-point that is not 3-arrow.*

6. **Questions.** Theorems 3.3 and 3.15 come close to giving necessary and sufficient conditions for a product to be an arrow ultrafilter. Nevertheless, they do fall short and this, of course, suggests the following:

QUESTION 6.1. Suppose U and V are incompatible arrow P-points. Is $U \times V$ then necessarily an arrow ultrafilter?

An affirmative answer to 6.1 would also provide an affirmative answer to the following.

QUESTION 6.2. Suppose $U \times V$ is an arrow ultrafilter. Is $V \times U$ then necessarily an arrow ultrafilter?

Kunen has shown [8] that it is consistent with ZFC that there are no Ramsey ultrafilters on ω, and Blass [2] has used this to show that it is also consistent with ZFC that there are no weakly-Ramsey ultrafilters on ω. This suggests several questions, of which we mention two. (The remark following the proof of Theorem 3.3 is also relevant here.)

QUESTION 6.3. Can one prove in ZFC that arrow ultrafilters exist?

QUESTION 6.4. Is it consistent with ZFC that there exists a unique (up to isomorphism) Ramsey ultrafilter on ω?

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